Second Order Conditions for Kuhn-Tucker Sufficient Optimality for Optimization Problems with Linear Matrix Inequality Constraints

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Abstract:
The Kuhn-Tucker Sufficiency Theorem states that a feasible point that satisfies the Kuhn-Tucker conditions is a global minimizer for a convex programming problem for which a local minimizer is global. In this paper, we present a new second order conditions for Kuhn-Tucker sufficiency for minimizing a smooth function with linear matrix inequality constraints and bounds on the variables. In particular, we provide a necessary and sufficient conditions for a local minimizer to be a global minimizer over a box when the objective function is weighted sum of squares and linear functions. Numerical examples are given to illustrate the significance of sufficiency criteria.

Keywords: Smooth nonlinear programming problems, linear matrix inequality constraints, Kuhn-Tucker conditions, sufficient global optimality.

1 Introduction
In this paper, we develop conditions under which a Kuhn-Tucker point is a global minimizer of a multi-extremal smooth mathematical semidefinite programming model problem:

\[(LIMP) \quad \min_{x \in \mathbb{R}^n} f(x) \]
\[s.t. \quad F_0 + \sum_{i=1}^n x_i F_i \succeq 0 \]
\[x_i \in [u_i, v_i]; \quad i = 1, 2, \ldots, n,\]

Where \(f: \mathbb{R}^n \to \mathbb{R}\) is twice continuously differentiable function on an open set containing a compact set \(\Delta = \{(x_1, \ldots, x_n)^T | x_i \in [u_i, v_i] \text{ for } i = 1, \ldots, n \}\), and \(u_i \leq v_i, \quad i = 1, \ldots, n, \quad F_i \in S^m, \quad i = 1, \ldots, m \) and \(S^m\) is the set of all symmetric \(m \times m\) matrices. The linear matrix inequality \(\text{(LMI)}\) constraint, \(F_0 + \sum_{i=1}^n x_i F_i \succeq 0\) means that the matrix \(F_0 + \sum_{i=1}^n x_i F_i\) is positive semidefinite. Optimization model problems with \(\text{(LMI)}\) constraints are also known as semidefinite optimization.

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problems (9,10). Model problems of the form (LIMP) cover large classes of nonconvex continuous optimization problems. Various generalized convexity conditions such as pseudo-convexity and quasi-convexity, just to name a few, have been given in the literature for a Kuhn-Tucker point to be a global minimizer of a nonlinear programming problem (1,4,8,11) and they often apply to problems where a local minimum is global. These Kuhn-Tucker sufficiency criteria have limited value for multi-extremal optimization problems.

The purpose of this paper is to present sufficient conditions for a given Kuhn-Tucker point to be a global minimizer of the general optimization model problem with (LMI) constraints; (LMIP). We obtain sufficient conditions for global optimality by constructing quadratic underestimators then by characterizing global minimizers of the underestimators. As a special case, we apply the results obtained to quadratic programming problems. In particular, we provide a necessary and sufficient conditions for a local minimizer to be a global minimizer over a box when the objective function is weighted sum of squares and linear functions. We discuss examples to illustrate the significance of the optimality conditions, presented in this paper.

2 Sufficient Global Optimality conditions

In this section we obtain sufficient global optimality conditions for smooth nonconvex minimization problem (LMIP) with linear matrix inequality constraints, by the method of underestimation.

We begin by presenting definitions and notations that will be used throughout this section. $F_i \in S^m, \ i = 1, \ldots, m$ and $S^m$ is the space of all symmetric $m \times m$ matrices.

Let $S^m_i = \{M \in S^m | M \succeq 0\}$

$$\Gamma = \{x \in \mathbb{R}^n | F_0 + \sum_{i=1}^{n} x_i F_i \in S^m_i\}$$

and

$$D := \Gamma \cap \Delta$$

Let,

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i, \quad \hat{F}(x) = \sum_{i=1}^{n} x_i F_i, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Then $\hat{F}(\ )$ is a linear operator from $\mathbb{R}^n$ to $S^m$ and its dual is defined by

$$\hat{F}^*(M) = (\text{tr}[F_1 M], \ldots, \text{tr}[F_n M])^T$$ for any $M \in S^m$,

where $\text{tr}[\ ]$ is the trace operation. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in D$. Define, for each $i=1,2,\ldots,n$. 

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If \( \bar{x} = (x_1, \ldots, x_n) \in D \) is a local minimizer of \((LIMP)\) and if a certain constraint qualification holds then the following Kuhn-Tucker conditions hold:

\[
(\exists M \in S^n_+ \text{ such that } \text{tr}[MF(\bar{x})] = 0) \text{ and } (\nabla f(\bar{x}) - \bar{F}^*(M))^T(x - \bar{x}) \geq 0, \forall x \in \Delta. \tag{2}
\]

The condition (2) can equivalently be written as

\[
(\exists M \in S^n_+ \text{ such that } \text{tr}[MF(\bar{x})] = 0) \text{ and } \bar{x}_i(\nabla f(\bar{x}) - \bar{F}^*(M))_i \leq 0, \forall i = 1, \ldots, n. \tag{3}
\]

We now state a second order sufficiency conditions for a Kuhn Tucker point to be a global minimizer of \((LIMP)\).

**Theorem 2.1** For \((LMI)\); let \( \bar{x} \in D = \Gamma \cap \Delta \) and the Kuhn-Tucker conditions hold at \( \bar{x} \) with multiplier \( M \in S^n_+ \). If, for each \( x \in D \),

\[ [SC] \quad \nabla^2 f(x) - \text{diag} \left( \frac{2\bar{x}_1 \nabla \left( f(\bar{x}) - \bar{F}^*(M) \right)}{v_1 - u_1}, \ldots, \frac{2\bar{x}_n \left( \nabla f(x) - \bar{F}^*(M) \right)_n}{v_n - u_n} \right) \succeq 0 \]

then \( \bar{x} \) is a global minimizer of \((LIMP)\).

**Proof.** Let \( Q = \text{diag} \left( \frac{2\bar{x}_1 \nabla \left( f(\bar{x}) - \bar{F}^*(M) \right)}{v_1 - u_1}, \ldots, \frac{2\bar{x}_n \left( \nabla f(x) - \bar{F}^*(M) \right)_n}{v_n - u_n} \right) \)

Define a quadratic function \( g: \mathbb{R}^n \to \mathbb{R} \) by

\[
g(x) := \frac{1}{2} x^T Q x + (\nabla f(\bar{x}) - Q(\bar{x}) - \bar{F}^*(M))^T x. \tag{4}
\]

Let \( l(x) := f(x) - \bar{F}^*(M)^T x - \text{tr}(MF_0), \ x \in \mathbb{R}^n \) and \( \phi(x) := l(x) - g(x), x \in \mathbb{R}^n \). Then it is easy to see that

\[ \nabla \phi(x) = 0 \text{ and } \nabla^2 \phi(x) = \nabla^2 f(x) - Q \geq 0 \quad \text{for all } x \in \Delta. \]

Therefore, \( \phi(x) - \phi(\bar{x}) \geq 0 \) for all \( x \in \Delta \), as \( \phi \) is a convex function.

Thus

\[
l(x) - l(\bar{x}) \geq g(x) - g(\bar{x}), \forall x \in \Delta \tag{5}
\]

Since \( M \in S^n_+ \) and \( F(x) \in S^n_+ \) for all \( x \in \Gamma \), we have \( \text{tr}[MF(x)] \geq 0 \), for all \( x \in D = \Delta \cap \Gamma \).

Hence, for all \( x \in D \),
We now show that (7) holds by considering following three cases.

Indeed, if there exists \( i_0 \) and \( x_{i_0} \) such that (7) is not fulfilled, by taking \( x = (\bar{x}_1, ..., \bar{x}_{i_0-1}, \bar{x}_{i_0}, \bar{x}_{i_0+1}, ..., \bar{x}_n) \), we have \( \bar{x} \in \Delta \) and

\[
g(x) - g(\bar{x}) = \sum_{i=1}^{n} \left( \frac{\bar{x}_i \nabla (f(\bar{x}) - \bar{F}^*(M))}{vi - ui} \right) (x_i - \bar{x}_i)^2 + \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_i (x_i - \bar{x}_i) < 0
\]

We now show that (7) holds by considering following three cases.

Case 1: \( \bar{x}_i = u_i \). Then, \( (\nabla f(\bar{x}) - \bar{F}^*(M))_i \geq 0 \). Thus,

\[
\sum_{i=1}^{n} \left( \frac{\bar{x}_i \nabla (f(\bar{x}) - \bar{F}^*(M))}{vi - ui} \right) (x_i - \bar{x}_i) + \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_i \geq 0, \forall x_i \in [u_i, v_i]
\]

Hence (7) holds.

Case 2: \( \bar{x}_i = v_i \). Then, \( (\nabla f(\bar{x}) - \bar{F}^*(M))_i \leq 0 \). Thus,

\[
\sum_{i=1}^{n} \left( \frac{\bar{x}_i \nabla (f(\bar{x}) - \bar{F}^*(M))}{vi - ui} \right) (x_i - \bar{x}_i) + \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_i \leq 0, \forall x_i \in [u_i, v_i]
\]

Therefore from (5), we have

\[
f(x) - f(\bar{x}) \geq g(x) - g(\bar{x}) \text{ for all } x \in D.
\]
Hence (7) holds.

**Case 3:** \( \bar{x}_i \in (u_i, v_i) \). Then, \( (\nabla f(\bar{x}) - \bar{F}^*(M))_i = 0 \) and clearly, (7) holds.

By combining all the above three cases, we have (7) holds and the conclusion follows.

For \( x \in D \), denote the eigenvalues of \( \nabla^2 f(x) \) by \( \delta(x), i = 1, 2, \ldots, n \). Let

\[
\delta^* = \min_{x \in D} \min_{i=1,\ldots,n} \delta_i(x)
\]

In the case where \( f \) is a quadratic function with the constant Hessian \( A \), \( \delta^* \) denotes the least eigen value of \( A \). We also observe that \( \delta^* \) is often used as a "convexifier" for "convexifying" a twice differentiable function by a quadratic term. In the following lemma we provide sufficient optimality conditions in terms of \( \delta^* \).

**Corollary 2.1** For \( (LMIP) \), let \( \bar{x} \in D = \Gamma \cap \Delta \). Suppose that there exist \( M \in S^n_+ \) such that \( \text{tr}[MF(\bar{x})] = 0 \) and \( \bar{x}_i(\nabla f(\bar{x}) - \bar{F}^*(M))_i \leq 0, \forall i = 1, \ldots, n \). If

\[
[S\bar{C}1] \quad \delta^* + \min_{i=1,\ldots,n} \left\{ \frac{2\bar{x}_i \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_i}{v_i - u_i} \geq 0 \right\}
\]

then \( \bar{x} \) is a global minimizer of \( (LMIP) \).

**Proof.** Let, \( \lambda(x) \) be the least eigen value of

\[
\nabla^2 f(x) - \text{diag} \left( \frac{2\bar{x}_i \nabla (f(\bar{x}) - \bar{F}^*(M))_i}{v_i - u_i}, \ldots, \frac{2\bar{x}_n \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_n}{v_n - u_n} \right).
\]

By the variational characterization of the least eigen value, we have,

\[
\lambda(x) \geq \min_{i=1,\ldots,n} \delta_i(x) + \min_{i=1,\ldots,n} \left( \frac{-2\bar{x}_i \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_i}{v_i - u_i} \right).
\]

Therefore, for each \( x \in D \),

\[
\lambda(x) \geq \min_{i=1,\ldots,n} \delta^*(x) + \min_{i=1,\ldots,n} \left( \frac{-2\bar{x}_i \left( \nabla f(\bar{x}) - \bar{F}^*(M) \right)_i}{v_i - u_i} \right) \geq 0
\]

Thus, for each \( x \in D \),
\[ \nabla^2 f(x) - \text{diag}\left( \frac{2 \tilde{X}_1 \nabla f(\tilde{x}) - \tilde{F}^*(M)}{v_1 - u_1}, ..., \frac{2 \tilde{X}_n \nabla f(x) - \tilde{F}^*(M)}{v_n - u_n} \right) \geq 0 \]

Hence, the conclusion follows from Theorem(2.1).

3 Quadratic minimization

In this section, we consider model problems \( LMI P \) with quadratic objective function:

\[
(LIMP - Q) \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x + a^T x
\]

\[
s.t \quad F_0 + \sum_{i=1}^{n} x_i F_i \geq 0
\]

\[
x_i \in [u_i, v_i]; \quad i = 1, 2, ..., n,
\]

where \( A = (a_{ij}) \in \mathbb{S}^n \) and \( a \in \mathbb{R}^n \).

**Theorem 3.1** For \((LMI P - Q)\), let \( \tilde{x} \in D \). Suppose that \( M \in \mathbb{S}^m_+ \) such that \( tr[MF(\tilde{x})] = 0 \) and \( \chi_i(A\tilde{x} + a - \tilde{F}^*(M)) \leq 0, \forall i = 1, ..., n \). Let \( \mu \) be the least eigen value of \( A \). If

\[
\mu + \min_{i=1, ..., n} \left( -2 \tilde{X}_i \frac{(A\tilde{x} + a - \tilde{F}^*(M))_i}{v_i - u_i} \right) \geq 0
\]

then \( \tilde{x} \) is a global minimizer of \((LMI P)\).

Proof: By the variational characterization of the least eigen value, we have

\[
\min_{v^T v = 1} v^T A v + \min_{i=1, ..., n} \left( -2 \tilde{X}_i \frac{(A\tilde{x} + a - \tilde{F}^*(M))_i}{v_i - u_i} \right) \geq 0
\]

Thus,

\[
\min_{v^T v = 1} v^T \left( A - \text{diag}\left( \frac{2 \tilde{X}_1 \left( A\tilde{x} + a - \tilde{F}^*(M) \right)_1}{v_1 - u_1}, ..., \frac{2 \tilde{X}_n \left( A\tilde{x} + a - \tilde{F}^*(M) \right)_n}{v_n - u_n} \right) \right) v \geq 0
\]

Hence the least eigen value of

\[
A - \text{diag}\left( \frac{2 \tilde{X}_1 \left( A\tilde{x} + a - \tilde{F}^*(M) \right)_1}{v_1 - u_1}, ..., \frac{2 \tilde{X}_n \left( A\tilde{x} + a - \tilde{F}^*(M) \right)_n}{v_n - u_n} \right) \geq 0
\]

Thus,
\[
A - \text{diag}\left(\frac{2\bar{x}_1 (A\bar{x} + a - \hat{f}^*(M))_1}{v_1 - u_1}, \ldots, \frac{2\bar{x}_n (A\bar{x} + a - \hat{f}^*(M))_n}{v_n - u_n}\right) \succeq 0
\]

Hence, the conclusion follows from Theorem 2.1.

**Theorem 3.2** For \((\text{LMIP} - Q)\), let \(\bar{x} \in D\). Suppose that \(M \in S_+^m\) such that \(\text{tr}[MF(\bar{x})] = 0\), and \(\chi_i(A\bar{x} + a - \hat{f}^*(M))_i \leq 0, \forall i = 1, \ldots, n\). If, for each \(i = 1, \ldots, n\),

\[
|a_{ij}| - \sum_{i \neq j: j=1}^{n} |a_{ij}| \geq \frac{-2\bar{x}_n (A\bar{x} + a - \hat{f}^*(M))_n}{v_n - u_n}
\]

then \(\bar{x}\) is a global minimizer of \((\text{LMIP})\).

**Proof.** Since, \(\chi_i(A\bar{x} + a - \hat{f}^*(M))_i \leq 0, \forall i = 1, \ldots, n\), (8) implies that,

\[
a_{ii} - \frac{2\bar{x}_i (A\bar{x} + a - \hat{f}^*(M))}{v_i - u_i} \geq |a_{ii}| - \left|\frac{2\bar{x}_i (A\bar{x} + a - \hat{f}^*(M))}{v_i - u_i}\right| = |a_{ii}| + \frac{2\bar{x}_i (A\bar{x} + a - \hat{f}^*(M))}{v_i - u_i}
\]

\[
\geq \sum_{i \neq j: j=1}^{n} |a_{ij}|
\]

Therefore matrix

\[
A - \text{diag}\left(\frac{2\bar{x}_1 (A\bar{x} + a - \hat{f}^*(M))_1}{v_1 - u_1}, \ldots, \frac{2\bar{x}_n (A\bar{x} + a - \hat{f}^*(M))_n}{v_n - u_n}\right)
\]

is diagonally dominant x and hence positive semi-definite. So, the conclusion follows from Theorem(2.1).

We now consider a special case of \((\text{LMIP} - Q)\) where the objective function is weighted sum of squares with \(A\) is a diagonal matrix, \(A = \text{diag}(a_{11}, \ldots, a_{nn})\) and without any constraints, over a box:

\[
(QW) \quad \min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \frac{1}{2} a_{ii}^2 x_i^2 + a_i x_i
\]

\(x_i \in [u_i, v_i]; \ i = 1, 2, \ldots, n,\)
We give a necessary and sufficient conditions for a Kuhn Tucker point \( \bar{x} \) to be a global minimizer of \((QW)\).

**Theorem 3.3** Let \( \bar{x} \) be a local minimizer of \((QW)\). Then, \( \bar{x} \) is a global minimizer of \((QW)\) if and only if

\[
a_{ii}(v_i - u_i) - 2\bar{x}_i(a_{ii}\bar{x}_i + a_i) \geq 0, \forall \ i = 1, \ldots, n. \tag{9}
\]

Proof. Suppose that

\[
a_{ii}(v_i - u_i) - 2\bar{x}_i(a_{ii}\bar{x}_i + a_i) \geq 0, \forall \ i = 1, \ldots, n.
\]

Then,

\[
A - \text{diag}(\frac{2\bar{x}_1(A\bar{x} + a)_1}{v_1 - u_1}, \ldots, \frac{2\bar{x}_n(A\bar{x} + a)_n}{v_n - u_n}) \succeq 0
\]

where \( A = \text{diag}(a_{11}, \ldots, a_{nn}) \). Thus, by applying Theorem 2.1, \( \bar{x} \) becomes a global minimizer of \((QW)\). Suppose that \( \bar{x} \) is a global minimizer of \((QW)\), then by Theorem 2.1 of (Jeyakumar, et al., 2006), it is necessary that,

\[
\frac{1}{2}\max\{0, -a_{ii}\} (v_i - u_i) + \bar{x}_i(a_{ii}\bar{x}_i + a_i) \leq 0, \forall i = 1, \ldots, n
\]

Hence,

\[
\frac{1}{2} - a_{ii}(v_i - u_i) + \bar{x}_i(a_{ii}\bar{x}_i + a_i) \leq 0, \forall i = 1, \ldots, n
\]

Therefore, \( a_{ii}(v_i - u_i) - 2\bar{x}_i(a_{ii}\bar{x}_i + a_i) \geq 0, \forall \ i = 1, \ldots, n \) and the conclusion follows.

We now apply Theorem 3.1 to the following Fractional programming Problem:

\[
(FP) \quad \min_{x \in \mathbb{R}^n} \frac{\sum_{i=1}^{n} \frac{1}{2} a_i^2 x_i^2 + b_i x_i}{\sum_{i=1}^{n} \frac{1}{2} c_i x_i^2 + d_i x_i}
\]

\( x_i \in [u_i, v_i]; \ i = 1, 2, \ldots, n, \)

where, \( a_b, b_1, c_b, d_1, u_b, v_b, v_i \in \mathbb{R}; \ i = 1, \ldots, n \), and, for each \( x \in D \), \( \sum_{i=1}^{n} \frac{1}{2} a_i x_i^2 + b_i x_i \geq 0 \) and \( \sum_{i=1}^{n} \frac{1}{2} c_i x_i^2 + d_i x_i \geq 0 \).

**Theorem 3.4** Let \( \bar{x} \) be a local minimizer of \((FP)\). Then, \( \bar{x} \) is a global minimizer of \((FP)\) if and only if

\[
(a_i - q(x)c_i)(v_i - u_i) - 2\bar{x}_i(a_i\bar{x}_i + b_i - q(x)(c_i\bar{x}_i + d_i)) \geq 0, \forall \ i = 1, \ldots, n.
\]

Proof. For \( x \in D \), define

\[
q(x) = \frac{\sum_{i=1}^{n} \frac{1}{2} a_i^2 x_i^2 + b_i x_i}{\sum_{i=1}^{n} \frac{1}{2} c_i x_i^2 + d_i x_i}
\]
Note that, $\bar{x}$ is a global minimizer/local minimizer of $(FP)$ if and only if $\bar{x}$ is a global minimizer/local minimizer of the following minimization of the form $(QW)$:

$$(FPW) \frac{\min}{x \in \mathbb{R}^n} \sum_{i=1}^n \frac{1}{2} a_i^2 x_i^2 + b_i x_i - q(x) \left( \sum_{i=1}^n \frac{1}{2} c_i^2 x_i^2 + dx_i \right)$$

By applying Theorem 3.1, $\bar{x}$ is a global minimizer of $(FPW)$ if and only if

$$(a_i - q(\bar{x})c_i)(v_i - u_i) - 2\bar{x}_i(a_i\bar{x}_i + b_i - q(\bar{x})(c_i\bar{x}_i + d_i)) \geq 0, \forall i = 1, \ldots, n.$$ 

Hence, the conclusion follows.

Finally, we apply Theorem 2.1 when inequality constraints are replaced by the standard linear inequalities:

$$(LIP) \min_{x \in \mathbb{R}^n} f(x) \quad s.t. \quad b_0 + Bx \geq 0 \quad x_i \in [u_i, v_i]; \quad i = 1, 2, \ldots, n,$$

where $B = (b_{ij})$ is an $m \times n$ matrix and $b_0 = (b_{01}, \ldots, b_{0m})^T, \lambda \in \mathbb{R}^n$.

**Corollary 3.1** Let $\bar{x} \in D$. Suppose that, $\lambda \in \mathbb{R}^n$, such that $\lambda^T (b_0 + Bx) = 0$ and $\chi_i(\nabla f(\bar{x}) - B\lambda)_i \leq 0, \forall i = 1, \ldots, n$. If, for each $x \in \Delta$,

$$\nabla^2 f(x) = \text{diag} \left( \frac{2\bar{x}_1 (A\bar{x} + a)_1}{v_1 - u_1}, \ldots, \frac{2\bar{x}_n (A\bar{x} + a)_n}{v_n - u_n} \right) \geq 0$$

then $\bar{x}$ is a global minimizer of $(LIP)$.

Proof. For each $i = 0, 1, \ldots, n$, let $F_i = \text{diag}(b_{i1}, \ldots, b_{im})$. Let $M = \text{diag}(\lambda)$. Then $\tilde{F}^*(M) = B\lambda$.

Then, by applying Theorem 2.1 the conclusion follows.

**Example 1** Consider the following smooth minimization model problem

$$\min_{x \in \mathbb{R}^2} f(x) = x_1^2 x_2 + x_1 x_2^2 - x_1 - x_2 \quad s.t \quad F_0 + \sum_{i=1}^2 x_i F_i \geq 0,$$

where

$$F_0 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\bar{x} = (\bar{x}_1, \bar{x}_2) = (0, 0) \in \Delta$. Clearly $\bar{x} \in D$.

We now check for $\bar{x} = (0, 0)$:

Let $z \in \Delta$. 

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Then \( F(\bar{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \nabla f(\bar{x}) = (-1, -1)^T \)

And \( \nabla^2 f(z) = \begin{pmatrix} 2x_2 & 2x_1 + 2x_2 \\ 2x_1 + 2x_2 & 2x_1 \end{pmatrix} \)

Taking \( M = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), we obtain \( M \in S^3_+, tr(MF(\bar{x})) = 0, \tilde{F}^*(M) = (4,2)^T \) and \( \bar{x}_i(\nabla f(\bar{x}) + \tilde{F}^*(M))_i \geq 0, \forall i = 1, \ldots, n. \)

\[ H_z = \nabla^2 f(z) - \text{diag}(2\bar{x}_1(\nabla f(\bar{x}) + \tilde{F}^*(M)))_1, \bar{x}_2(\nabla f(\bar{x}) + \tilde{F}^*(M))_2 \]

\[ = \begin{pmatrix} 2x_2 + 10 & 2x_1 + 2x_2 \\ 2x_1 + 2x_2 & 2x_1 + 6 \end{pmatrix} \]

Now \((2z_2 + 10) > 0, \forall (z_1, z_2) \in \Delta \)

\[ \det \begin{pmatrix} 2x_2 + 10 & 2x_1 + 2x_2 \\ 2x_1 + 2x_2 & 2x_1 + 6 \end{pmatrix} = 60 = 20z_1 + 12z_2 + 4z_1z_2 - 4z_1^2 - 4z_2^2 \]

\[ > 0, \forall (z_1, z_2) \in \Delta \]

Therefore \( H_z \) is positive semidefinite for each \((z_1, z_2) \in \Delta. \) Hence we see that \([SC] \) is satisfied at \( \bar{x} = (0,0) \) which is a global minimizer of \((E2)\).

**Example 2** Consider the following smooth minimization model problem \((QW)\):

\[ (E2) \min_{x \in \mathbb{R}^3} f(x) = -x_1^2 + x_2^2 - 3x_3 - x_1 - x_2, \]

\[ s.t \ x \in \Delta = [-1,1] \times [-1,1] \times [-1,1] \]

\((-1,0,-1), (-1,0,1), (1,0,-1), (1,0,1) \) are the local minimizers of \( E(2) \). It is easy to check that, the condition \((9) \) is satisfied only at \( \bar{x} = (1,0,1) \) and \( \bar{x} \) is indeed the global minimizer of \( E(2) \)

**References**


